

On the Average Order of a Class of Arithmetical Functions*

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We consider a class of arithmetical functions generated by Dirichlet series that satisfy a functional equation with multiple gamma factors. We prove one- and two-sided omega theorems for the error terms associated with summatory functions of the type $\sum_{\lambda_n \leq x} a(n)(x - \lambda_n)^\rho$, where $\rho \geq 0$. In particular, we improve results of Hardy for the circle and Dirichlet divisor problems and results of Szegő and Walfisz for the Piltz divisor problem in algebraic number fields.

Contents. 1. *Statement of the theorems* 1.1. General notation. 1.2. The theorems 2. *Preliminary results.* 2.1. Some well-known results. 2.2. Elementary estimates. 2.3. Analytical estimates. 3. *The Borel mean value of the error term.* 3.1. An exact formula. 3.2. An asymptotic formula. 4. *Proofs of the theorems.* 4.1. Theorem A. 4.2. Theorem B. 4.3. Theorem C. 5. *Examples.* 5.1. Piltz divisor problem. 5.2. Lattice points in ellipsoids. 5.3. Another divisor function.

INTRODUCTION

We consider a class of arithmetical functions generated by Dirichlet series that satisfy a functional equation with gamma factors of the type originally studied by Chandrasekharan and Narasimhan [4]. We prove one- and two-sided omega theorems for the error terms associated with summatory functions of the type

$$\sum_{\lambda_n \leq x} a(n)(x - \lambda_n)^\rho,$$

where $\rho \geq 0$. In particular, we obtain new results for the problem of determining the number of lattice points in a circle and in the Dirichlet and Piltz divisor problems. These improve results of Hardy [10] and Szegő and Walfisz [16, 17].

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In [4], Chandrasekharan and Narasimhan obtained both Ω - and O -results. For a more restrictive class, Berndt [2, 3] obtained better one- and two-sided Ω -results than Chandrasekharan and Narasimhan, although he considered only the case $\rho = 0$. In [1] Berndt stated, without proof, results for $\rho \geq 0$. We should remark that Berndt's proof in [2, 3] applied only to the case where the Dirichlet series was of the form $\sum_{n=1}^{\infty} a(n) \lambda_n^{-s}$ and

$$Ax < \sum_{\lambda_n \leq x} 1 < Bx \quad (1)$$

for some positive constants A and B and all sufficiently large x . Nonetheless, the applications to the classical problems that he indicated are correct. (In his examples involving positive definite quadratic forms, the restriction must be added that the quadratic form have rational coefficients.)

We consider here a more general class than Berndt, one that does not require (1). Our proof is based on ideas used by Szegő [15] to obtain Ω -results for the error term in the problem of determining the number of lattice points in a k -dimensional sphere, the so-called k -squares problem. These same ideas were used by Szegő and Walfisz [16, 17] to prove Hardy's conjecture [10] for the Piltz divisor problem in algebraic number fields. (Of course, $\rho = 0$ in these papers.) We should remark that there is an error in Szegő and Walfisz's proof of their Lemma 5 [17, p. 479]. This involved a circular argument in the application of Dirichlet's approximation theorem and in the choice of some parameters. (This same mistake was made by Berndt in [3].) It can, however, be corrected and is done so here. See Lemma 4.2.1.

In this paper we also improve the basic method to allow, in any given example, particular distribution properties of the given arithmetical function to be used. It is this improvement which leads to the new results in the classical problems as mentioned above.

1. STATEMENT OF THE THEOREMS

1.1. General Notation

We begin by specifying the class of arithmetical functions which we are considering.

DEFINITION 1.1.1. Let $\{a(n)\}$ and $\{b(n)\}$ be two sequences of complex numbers, not identically zero. Let $\{\lambda_n\}$ and $\{\mu_n\}$ be two strictly increasing sequences of positive numbers tending to ∞ . Suppose the series

$$\phi(s) = \sum_{n=1}^{\infty} a(n) \lambda_n^{-s}$$

and

$$\psi(s) = \sum_{n=1}^{\infty} b(n) \mu_n^{-s}$$

converge in some half-plane and have abscissas of absolute convergence σ_a^* and σ_a , respectively. For each $v = 1, 2, \dots, N$ suppose that $\alpha_v > 0$ and β_v is complex, and let

$$\Delta(s) = \prod_{v=1}^N \Gamma(\alpha_v s + \beta_v).$$

If r is a real number, we say that ϕ and ψ satisfy the functional equation

$$\Delta(s) \phi(s) = \Delta(r-s) \psi(r-s)$$

if there exists in the s -plane a domain D that is the exterior of a compact set S and on which there exists a holomorphic function χ such that

$$(i) \quad \lim_{|t| \rightarrow \infty} \chi(\sigma + it) = 0,$$

uniformly in every interval $-\infty < \sigma_1 \leq \sigma \leq \sigma_2 < \infty$; and

$$(ii) \quad \begin{aligned} \chi(s) &= \Delta(s) \phi(s) & \text{for } \sigma > \sigma_a^* \\ &= \Delta(r-s) \psi(r-s) & \text{for } \sigma < r - \sigma_a. \end{aligned}$$

This definition was first given by Chandrasekharan and Narasimhan [4].

Let ρ be a fixed nonnegative real number, and define the following quantities:

$$\begin{aligned} \alpha &= \sum_{v=1}^N \alpha_v, \\ \theta &= \theta_\rho = r/2 - 1/(4\alpha) + \rho(1 - 1/(2\alpha)), \\ \kappa &= \kappa_\rho = \sigma_a - r/2 - 1/(4\alpha) - \rho/(2\alpha), \\ \mu &= \frac{1}{2} + \sum_{v=1}^N (\beta_v - \frac{1}{2}), \\ \beta &= \beta_\rho = -(\mu + \alpha r/2 + \rho/2 + 1/4) \\ \varepsilon_\rho &= \text{sgn}(\cos \beta_\rho \pi) \\ b &= b_\rho = \beta - [\beta + \frac{1}{2}] \\ \gamma &= \gamma_\rho = 2\alpha\kappa - 1. \end{aligned} \tag{1.1.1}$$

We are assuming here that μ is real so that β is real also.

For $x > 0$, define the summatory function $A_\rho(x)$ by

$$A_\rho(x) = \Gamma(\rho + 1)^{-1} \sum'_{\lambda_n \leq x} a(n)(x - \lambda_n)^\rho, \quad (1.1.2)$$

where the prime on the summation sign indicates the last term is to be multiplied by $\frac{1}{2}$ if $x = \lambda_n$ for some n . (This notation will be used throughout the sequel.)

Let L be such that $L > |r/2 - 1/(4\alpha)|$ and all the singularities of $\phi(s)$ lie in $\sigma > -L$. Then define $Q_\rho(x)$, a residual function, for $x > 0$ by

$$Q_\rho(x) = \frac{1}{2\pi i} \int_{\mathcal{C}_\rho} \frac{\phi(s) \Gamma(s) x^{s+\rho}}{\Gamma(s+\rho+1)} ds, \quad (1.1.3)$$

where \mathcal{C}_ρ is any set of curves enclosing the singularities of the integrand to the right of $\sigma = -\rho - 1 - L$. We define the error term $P_\rho(x)$ by

$$P_\rho(x) = A_\rho(x) - Q_\rho(x). \quad (1.1.4)$$

Similarly, let $Q_\phi(x)$ be defined for $x > 0$ by

$$Q_\phi(x) = \frac{1}{2\pi i} \int_{\mathcal{C}^*} \frac{\psi(s) x^s}{s} ds, \quad (1.1.5)$$

where \mathcal{C}^* is any set of curves enclosing the singularities of the integrand. Put

$$B(x) = \sum'_{\mu_n \leq x} b(n) \quad (1.1.6)$$

and

$$P_\phi(x) = B(x) - Q_\phi(x). \quad (1.1.7)$$

Furthermore, we introduce notation for a set of “distinguished” integers. Let \mathcal{P} be any set of positive integers such that

$$B^*(x) = B^*(x; \mathcal{P}) = \sum'_{\substack{\mu_n \leq x \\ n \in \mathcal{P}}} b(n) \sim B(x), \quad (1.1.8)$$

as x tends to ∞ . We define a function $W(x)$ to serve as an approximate counting function for the set \mathcal{P} . For $x > 0$, let $W(x)$ be a strictly increasing, continuous function such that

$$W(x) \geq \# \{n : \mu_n \leq x, n \in \mathcal{P}\}. \quad (1.1.9)$$

Let

$$Z(x) = W^{-1}(x), \quad (1.1.10)$$

the inverse function of W . We make the following assumptions on Z (and W):

(i) For every positive constant A , there exist positive constants $c_1(A)$ and $c_2(A)$, depending only on A , such that

$$c_1(A) Z(x) \leq Z(Ax) \leq c_2(A) Z(x).$$

(ii) For some $\eta < \min\{1/2, 1/(3\alpha)\}$, and all $\tau > 0$,

$$Z(x + \tau) - Z(x) = O(\tau e^{\eta x})$$

as x tends to ∞ .

Note that condition (i) implies that $Z(x)$ has at most polynomial growth. Thus the following conditions are satisfied:

(iii) For some $\eta' > 0$,

$$Z(x) = O(x^{\eta'}),$$

as x tends to ∞ , and

$$(iii)' \quad \log Z(x) = O(\log x),$$

as x tends to ∞ .

We remark that if \mathcal{P} is the set of all positive integers, then (1.1.8) holds trivially, and $W(x)$ is essentially a continuous approximation to the counting function for the μ_n 's. This function is introduced primarily to extend Berndt's results to more general Dirichlet series. (See the remarks surrounding (1).) Furthermore, it is generally more effective to use a significantly thinner set \mathcal{P} .

Finally, before stating our theorems, we establish some notational conventions. The letter c will always denote a positive absolute constant, not necessarily the same at each occurrence. If the letter c appears with a sub- or superscript or with a prime, then its value is to remain fixed throughout. Also, the parameter ξ_0 will always denote a positive number taken sufficiently small to make the given estimate valid.

The symbol \sum will always denote the sum from $n = 1$ to ∞ . For $R > 0$ and for real numbers a and b , we denote by $\mathcal{C}(a, b)$ the oriented polygonal path with vertices $a - i\infty$, $a - iR$, $b - iR$, $b + iR$, $a + iR$, and $a + i\infty$, in that order. We also assume that all closed paths in the plane are taken in the positive or counterclockwise direction.

1.2. The Theorems

Assume the following conditions are satisfied. First, $b(n) \geq 0$ for all n . This is the main condition which restricts our class of arithmetical functions from that considered by Chandrasekharan and Narasimhan [4]. Berndt [1-3] also made this assumption.

We also assume that there exist constants c_0 and l so that

$$B(x) \sim Q_\psi(x) \sim c_0 x^{\sigma_a} (\log x)^{l-1}, \quad (1.2.1)$$

as x tends to ∞ . We remark that (1.2.1) is satisfied for $b(n) \geq 0$ if $r > 0$ and all the singularities of ψ are poles. (See Berndt [2, p. 186] and Chandrasekharan and Narasimhan [4, Theorem 4.1, p. 111].)

It is clear from (1.2.1) that both

$$P_\psi(x) = o(x^{\sigma_a} (\log x)^{l-1}), \quad (1.2.2)$$

and

$$B^*(x) \sim c_0 x^{\sigma_a} (\log x)^{l-1}, \quad (1.2.3)$$

as x tends to ∞ .

For notational convenience we define a function $g(x)$. For $x > 0$, let

$$\begin{aligned} g(x) &= \{Z(|\log x|)\}^\kappa \{\log Z(|\log x|)\}^{l-1} & \text{if } \kappa > 0 \\ &= \{\log Z(|\log x|)\}^l & \text{if } \kappa = 0, \end{aligned} \quad (1.2.4)$$

where $Z(x)$ and $\kappa = \kappa_\rho$ are defined in (1.1.10) and (1.1.1), respectively.

We can now state our one-sided omega theorem. (Recall that $\theta = \theta_\rho$ is given in (1.1.1).)

THEOREM A. *With the above notation and assumptions, if $\kappa \geq 0$ and $\varepsilon_\rho \neq 0$, then*

$$\varepsilon_\rho \operatorname{Re} P_\rho(x) = \Omega_+(x^\theta g(x)).$$

The case $\kappa < 0$ is completely known and can be found in Chandrasekharan and Narasimhan [4, Theorem 3.1].

For Theorems B and C (the two-sided omega theorems), we drop the condition that $\varepsilon_\rho \neq 0$, but, of course, we require certain other conditions on ρ be satisfied. (These are given in the statement of Theorem B.) Also we require the following two technical assumptions, in addition to those above for Theorem A.

First, we assume that as x tends to ∞ ,

$$Q'_\psi(x) \sim c_0 \sigma_a x^{\sigma_a-1} (\log x)^{l-1}, \quad (1.2.5)$$

where c_0 and l are given above in (1.2.1).

Secondly, we assume that there exists a constant $T > 0$ so that as x tends to ∞ ,

$$A_0(x) = O(x^T). \quad (1.2.6)$$

We note that these assumptions are certainly satisfied if all the singularities of ϕ and ψ are poles.

THEOREM B. *With all the above notations and assumptions, if $\kappa > 0$ and either*

$$2b > -\gamma \quad \text{or} \quad 2b < \gamma, \quad (1.2.7)$$

then

$$\mathbf{Re} P_\rho(x) = \Omega_\pm(x^\theta g(x)).$$

THEOREM C. *Under the same hypotheses as Theorem B, there exist positive constants A and B and a strictly increasing sequence $\{y_n\}$ tending to ∞ such that the two inequalities*

$$\pm \mathbf{Re} P_\rho(x) > Ax^\theta g(x)$$

have solutions in each interval

$$y_n \leq x \leq y_n + By_n^{1-1/(2\alpha)} (\log y_n)^{1/2} \{Z(\log y_n)\}^{-1/(2\alpha)}.$$

Conditions (1.2.7) are necessary to insure that a certain function ($h(a)$ defined in (2.3.27)) has a change of sign. Since $b = b_\rho$ and $\gamma = \gamma_\rho$, (1.2.7) reduces to a restriction on the range of ρ . (The parameters b and γ are given in (1.1.1).)

We should also remark that if β is not real, then similar results can be given for $\mathbf{Im} P_\rho(x)$ as well. This requires only a slight, though notationally inconvenient, change in the proof.

2. PRELIMINARY RESULTS

2.1. Some Well-Known Results

In this section we collect some classical results which will be used in the sequel. These are all given without proof.

First, we require the following forms of Stirling's formula. (They will be cited in the sequel by their corresponding letter.)

(A) For real numbers σ and t ,

$$|\Gamma(\sigma + it)| = \sqrt{2\pi} |t|^{\sigma-1/2} e^{-\pi|t|/2} \{1 + O(|t|^{-1})\}$$

as $|t|$ tends to ∞ , uniformly in σ in any interval $-\infty < \sigma_1 \leq \sigma \leq \sigma_2 < \infty$.

(B) For a real number τ ,

$$\Gamma(n + \tau) = \sqrt{2\pi} n^{n+\tau-1/2} e^{-n} \{1 + O(n^{-1})\},$$

as n tends to ∞ .

(C) For a real number τ ,

$$\Gamma(n + \tau)/\Gamma(n) = n^\tau \{1 + O(n^{-1})\},$$

as n tends to ∞ .

The next lemma is fundamental to the method. It is called Dirichlet's approximation theorem.

LEMMA 2.1.1. *Let $\gamma_1, \gamma_2, \dots, \gamma_M$ be M fixed positive numbers. Let $m > 0$, and let a positive integer q be given. Then there exist a t in the interval $|m, q^M m|$ and positive integers y_1, y_2, \dots, y_M such that*

$$|t\gamma_n - y_n| < 1/q,$$

for $n = 1, 2, \dots, M$.

2.2. Elementary Estimates

This section contains some elementary lemmas involving certain summatory functions of the arithmetical functions $a(n)$ and $b(n)$. In all but the last lemma, we assume only the hypotheses of Theorem A. In the last lemma, Lemma 2.2.7, we assume also (1.2.6).

The first three lemmas can be proved easily from (1.2.1), (1.2.3), and summations by parts.

LEMMA 2.2.1. *For $\tau \leq \sigma_a$, as x tends to ∞ ,*

$$\begin{aligned} \sum_{\mu_n \leq x}' b(n) \mu_n^{-\tau} &\sim c x^{\sigma_a - \tau} (\log x)^{l-1} & \text{if } \tau < \sigma_a \\ &\sim c (\log x)^l & \text{if } \tau = \sigma_a. \end{aligned}$$

LEMMA 2.2.2. *For $\tau \leq \sigma_a$, as x tends to ∞ ,*

$$\begin{aligned} \sum_{\substack{\mu_n \leq x \\ n \notin \mathcal{P}}} b(n) \mu_n^{-\tau} &= o(x^{\sigma_a - \tau} (\log x)^{l-1}) & \text{if } \tau < \sigma_a \\ &= o((\log x)^l) & \text{if } \tau = \sigma_a. \end{aligned}$$

LEMMA 2.2.3. For $\tau > \sigma_a$, as x tends to ∞ ,

$$\sum_{\mu_n > x} b(n) \mu_n^{-\tau} = O(x^{\sigma_a - \tau} (\log x)^{l-1}).$$

As an immediate consequence of Lemma 2.2.1, we have, for each $\varepsilon' > 0$ and each $\tau \leq \sigma_a$,

$$\sum_{\mu_n \leq x} b(n) \mu_n^{-\tau} = O(x^{\sigma_a - \tau + \varepsilon'}). \quad (2.2.1)$$

Next define for $x > 0$ and $\xi > 0$,

$$E(x) = \exp \{ -(\xi x)^{1/\alpha} \}. \quad (2.2.2)$$

LEMMA 2.2.4. For $\tau \leq \sigma_a$, as ξ tends to zero,

$$\begin{aligned} \sum b(n) E(\mu_n) \mu_n^{-\tau} &\geq c \xi^{-(\sigma_a - \tau)} (\log 1/\xi)^{l-1} && \text{if } \tau < \sigma_a \\ &\geq c (\log 1/\xi)^l && \text{if } \tau = \sigma_a. \end{aligned}$$

Proof. For $\tau \leq \sigma_a$, let

$$B_1(x) = \sum_{\mu_n \leq x} b(n) \mu_n^{-\tau}.$$

Then, since $b(n) \geq 0$,

$$\begin{aligned} \sum b(n) E(\mu_n) \mu_n^{-\tau} &= \int_{\mu_1}^{\infty} E(u) dB_1(u) \\ &= \frac{\xi^{1/\alpha}}{\alpha} \int_{\mu_1}^{\infty} B_1(u) u^{1/\alpha - 1} E(u) du \\ &\geq \frac{\xi^{1/\alpha}}{\alpha} \int_{1/\xi}^{\infty} B_1(u) u^{1/\alpha - 1} E(u) du \\ &\geq B_1(1/\xi) \frac{\xi^{1/\alpha}}{\alpha} \int_{1/\xi}^{\infty} u^{1/\alpha - 1} E(u) du \\ &\geq c B_1(1/\xi), \end{aligned}$$

where we have written $v = \xi u$ in the last integral. The lemma then follows easily from Lemma 2.2.1.

LEMMA 2.2.5. For $\tau \leq \sigma_a$, as ξ tends to zero,

$$\begin{aligned} \sum_{\substack{n=1 \\ n \notin \mathcal{P}}}^{\infty} b(n) E(\mu_n) \mu_n^{-\tau} &= o(\xi^{-(\sigma_a - \tau)} (\log 1/\xi)^{l-1}) \quad \text{if } \tau < \sigma_a \\ &= o((\log 1/\xi)^l) \quad \text{if } \tau = \sigma_a. \end{aligned}$$

Proof. We outline the main ideas of the proof for $\tau < \sigma_a$. The proof for $\tau = \sigma_a$ is similar. Set

$$B_2(x) = \sum_{\substack{\mu_n \leq x \\ n \notin \mathcal{P}}} b(n) \mu_n^{-\tau}.$$

By Lemma 2.2.2 (since $\tau < \sigma_a$), as x tends to ∞ ,

$$B_2(x) = o(x^{\sigma_a - \tau} (\log x)^{l-1}).$$

Thus, as ξ tends to zero,

$$\begin{aligned} \sum_{\substack{n=1 \\ n \notin \mathcal{P}}}^{\infty} b(n) E(\mu_n) \mu_n^{-\tau} &= \int_{\mu_1}^{\infty} E(u) dB_2(u) \\ &= \frac{\xi^{1/\alpha}}{\alpha} \int_{\mu_1}^{\infty} B_2(u) u^{1/\alpha - 1} E(u) du \\ &= o \left(\xi^{1/\alpha} \int_{\mu_1}^{\infty} u^{\sigma_a - \tau + 1/\alpha - 1} (\log u)^{l-1} E(u) du \right) \\ &= o \left(\xi^{-(\sigma_a - \tau)} \int_0^{\infty} v^{\sigma_a - \tau + 1/\alpha - 1} |\log v/\xi|^{l-1} \exp(-v^{1/\alpha}) dv \right) \\ &= o(\xi^{-(\sigma_a - \tau)} (\log 1/\xi)^{l-1}), \end{aligned}$$

where in the penultimate step we have made the change of variables $v = \xi u$. We note that this “proof” can be made rigorous with an appropriate epsilon argument.

Next, for $\xi > 0$ and $x > 0$, let

$$F(\xi, x) = \sum_{\mu_n > x} b(n) E(\mu_n) \mu_n^{-(\sigma_a - \kappa)}. \quad (2.2.3)$$

LEMMA 2.2.6. For $\kappa \geq 0$ and $x \geq \mu_1/2 > 0$, as ξ tends to zero,

$$\begin{aligned} F(\xi, x) &\ll \xi^{-\kappa} (\log 1/\xi)^{l-1} E(x/2^\alpha) & \text{if } \kappa > 0 \\ &\ll (\log 1/\xi)^l E(x/2^\alpha) & \text{if } \kappa = 0. \end{aligned}$$

Proof. We proceed as in the proof of Lemma 2.2.4 with $\tau = \sigma_a - \kappa$ in $B_1(x)$. By Lemma 2.2.1,

$$\begin{aligned} F(\xi, x) &= \int_{x^-}^{\infty} E(u) dB_1(u) \\ &= -E(x) B_1(x) + \frac{\xi^{1/\alpha}}{\alpha} \int_x^{\infty} B_1(u) u^{1/\alpha-1} E(u) du \\ &\ll \xi^{1/\alpha} \int_x^{\infty} u^{\kappa+1/\alpha-1} (\log u)^{l-1} E(u) du & \text{if } \kappa > 0 \\ &\ll \xi^{1/\alpha} \int_x^{\infty} u^{1/\alpha-1} (\log u)^l E(u) du & \text{if } \kappa = 0. \end{aligned}$$

We prove the estimate for $\kappa = 0$, as the proof for $\kappa > 0$ is similar. We assume $\xi \leq 1/e^2$.

First if $x \geq e^2/\xi$, then letting $v = \xi u$ in the last integral above, we find that

$$\begin{aligned} F(\xi, x) &\ll \xi^{1/\alpha} E(x/2^\alpha) \int_{e^2/\xi}^{\infty} u^{1/\alpha-1} (\log u)^l E(u/2^\alpha) du \\ &\ll E(x/2^\alpha) \int_{e^2}^{\infty} v^{1/\alpha-1} (\log v/\xi)^l \exp(-\tfrac{1}{2}v^{1/\alpha}) dv \\ &\ll (\log 1/\xi)^l E(x/2^\alpha), \end{aligned}$$

where we have used the inequality

$$\log v/\xi \leq (\log v)(\log 1/\xi),$$

which is valid since $v \geq e^2$ and $1/\xi \geq e^2$.

Next, if $x < e^2/\xi$, we divide the integral into two parts and estimate each part separately:

$$F(\xi, x) \ll \xi^{1/\alpha} E(x/2^\alpha) \left\{ \int_x^{e^2/\xi} + \int_{e^2/\xi}^{\infty} \right\} u^{1/\alpha-1} (\log u)^l E(u/2^\alpha) du.$$

The second integral is estimated exactly as above. In the first integral $E(u/2^\alpha) \leq 1$ and $\log u$ is increasing. Thus

$$\begin{aligned} \int_x^{e^{2/\xi}} u^{1/\alpha-1} (\log u)^l E(u/2^\alpha) du &\leq (\log 1/\xi)^l \int_0^{e^{2/\xi}} u^{1/\alpha-1} du \\ &\ll \xi^{-1/\alpha} (\log 1/\xi)^l. \end{aligned}$$

Combining these estimates completes the proof.

As consequences of the last two lemmas, the following results are valid:

If $\kappa \geq 0$ and $\varepsilon' > 0$, then as ξ tends to zero,

$$F(\xi, x) \ll \xi^{-\kappa-\varepsilon'} E(x/2^\alpha), \quad (2.2.4)$$

uniformly for $x > \mu_1/2$.

If $\kappa > 0$, there exists a constant c_1 so that if $0 < \xi < \xi_0$, then

$$F(\xi, x) \leq c_1 \xi^{-\kappa} (\log 1/\xi)^{l-1} E(x/2^\alpha) \quad (2.2.5)$$

uniformly in $x \geq \mu_1/2$ and

$$\sum'_{\substack{\mu_n \leq x \\ n \notin \mathcal{P}}} b(n) E(\mu_n) \mu_n^{-(\sigma_a - \kappa)} \leq c_1 \xi^{-\kappa} (\log 1/\xi)^{l-1} \quad (2.2.6)$$

uniformly in $X > 0$.

Similarly, if $\kappa \geq 0$, as ξ tends to zero,

$$\begin{aligned} \sum'_{\substack{\mu_n \leq X \\ n \notin \mathcal{P}}} b(n) E(\mu_n) \mu_n^{-(\sigma_a - \kappa)} &= o(\xi^{-\kappa} (\log 1/\xi)^{l-1}) \quad \text{if } \kappa > 0 \\ &= o((\log 1/\xi)^l) \quad \text{if } \kappa = 0, \end{aligned} \quad (2.2.7)$$

uniformly in $X > 0$.

LEMMA 2.2.7. Assume (1.2.6). Then as x tends to ∞ ,

$$P_\rho(x) = O(x^{T+\rho}).$$

Proof. We have

$$\begin{aligned} P_\rho(x) &= O(|A_\rho(x)|) \\ &= O\left(\left|\int_{\lambda_1^-}^x (x-u)^\rho dA_0(u)\right|\right) \\ &= O(|A_0(x)| x^\rho) \\ &= O(x^{T+\rho}), \end{aligned}$$

as claimed.

2.3. Analytical Estimates

We are assuming that $\rho \geq 0$ is fixed and that $\kappa \geq 0$. Let $\delta > 0$ and $a \geq 0$ be defined by

$$\begin{aligned}\delta^{-1} &= \max\{25, 8\alpha|\sigma_a - r - \rho| + 1, 16\alpha\kappa\}, \\ a &= \max\{0, -2\alpha\theta\}.\end{aligned}\tag{2.3.1}$$

Each of the conditions on δ and a are used to estimate certain quantities. Also, in all the following lemmas we assume only the hypotheses of Theorem A unless otherwise specified.

Let

$$c_\rho = r/2 + \rho/(2\alpha) - \varepsilon_0,\tag{2.3.2}$$

where $0 < \varepsilon_0 < 1/(4\alpha)$. Furthermore, let

$$c_\delta = r + \rho + (1 + a)/(2\alpha\delta) + \varepsilon_1,\tag{2.3.3}$$

where ε_1 is chosen so that $0 < \varepsilon_1 < 1/(4\alpha)$ and $c_\delta - r$ is not a nonnegative integer. Let R be a positive number with

$$R > \sup\{|\beta_v/\alpha_v|, v = 1, 2, \dots, N; |s|, s \in S\},\tag{2.3.4}$$

where S is the "singularity set" in Definition 1.1.1.

For any nonnegative integer q and for $x > 0$, define

$$I_{\rho+q}(x) = \frac{1}{2\pi i} \int_{\mathcal{C}(c_\rho, c_\delta)} G_{\rho+q}(s) x^{r+\rho+q-s} ds,\tag{2.3.5}$$

where

$$G_{\rho+q}(s) = \frac{\Gamma(r-s) \Delta(s)}{\Gamma(r+\rho+q+1-s) \Delta(r-s)}.\tag{2.3.6}$$

By our choice of c_ρ and Stirling's formula (A), we see that $I_{\rho+q}(x)$ is well defined for every q and converges uniformly in $q \geq 0$. Also, the choice of c_δ insures that the path $\mathcal{C}(c_\rho, c_\delta)$ does not pass through any of the poles of $G_{\rho+q}(s)$. Furthermore, the poles of $\Delta(s)$ are all to the left of the path.

LEMMA 2.3.1. *For each integer $q \geq 0$ and for $x > 0$,*

$$\frac{d}{dx} I_{\rho+q+1}(x) = I_{\rho+q}(x).$$

Proof. Since the path of integration is independent of q and $I_{\rho+q}(x)$

converges uniformly in $q \geq 0$ and x in any bounded set, we may differentiate under the integral sign. The result is then immediate from the identity $\Gamma(s+1) = s\Gamma(s)$.

LEMMA 2.3.2. *For any nonnegative integer m ,*

$$I_\rho(x) = \sum_{v=0}^m e_v x^{\theta-v/(2\alpha)} \cos(cx^{1/(2\alpha)} + k_v \pi) \\ + O(x^{\theta-(m+1)/(2\alpha)}) + O(x^{-(1+a)/(2\alpha\delta)}),$$

where the e_v are constants, $e_0 > 0$ and $k_v = \beta + v/2$.

Proof. We deform the path $\mathcal{C}(c_\rho, c_\delta)$ into a path $\mathcal{C}(c_\rho, c^*)$, where c^* is a certain constant depending on m and $c^* > c_\delta$. (In fact, we take $c^* = (m+1)/(2\alpha) - (\mu + \beta)/\alpha + \varepsilon_2$, where ε_2 is a certain positive number.) By Cauchy's theorem,

$$I_\rho(x) = \frac{1}{2\pi i} \int_{\mathcal{C}(c_\rho, c^*)} G_\rho(s) x^{r+\rho-s} ds \\ + \sum_{\substack{v \geq 0 \\ c_\delta < r+v < c^*}} \operatorname{res}_{s=r+v} G_\rho(s) x^{r+\rho-s}. \quad (2.3.7)$$

Since c_δ and c^* are constants, and by the definition of c_δ ,

$$\sum_{\substack{v \geq 0 \\ c_\delta < r+v < c^*}} \operatorname{res}_{s=r+v} G_\rho(s) x^{r+\rho-s} = O(x^{r+\rho-c_\delta}) \\ = O(x^{-(1+a)/(2\alpha\delta)}). \quad (2.3.8)$$

In the remainder of the proof, the following asymptotic formula for the integral on the right-hand side of (2.3.7) is developed:

$$\frac{1}{2\pi i} \int_{\mathcal{C}(c_\rho, c^*)} G_\rho(s) x^{r+\rho-s} ds = \sum_{v=0}^m e_v x^{\theta-v/(2\alpha)} \cos(cx^{1/(2\alpha)} + k_v \pi) \\ + O(x^{\theta-(m+1)/(2\alpha)}). \quad (2.3.9)$$

For a proof of (2.3.9) we refer the reader to Chandrasekharan and Narasimhan [5, pp. 33–36]. We should point out that in their proof, ρ is a nonnegative integer. However, the computation in (2.3.7) and (2.3.8) allows the generalization of the proof to nonintegral ρ . A proof of the lemma in complete detail can also be found in [7]. Inserting (2.3.8) and (2.3.9) in (2.3.7) completes the proof of the lemma.

For $x > 0$, let

$$M_k^\rho(x) = \frac{1}{\Gamma(k+1)} \int_0^\infty e^{-u} u^k I_\rho((2xu)^\alpha) du, \quad (2.3.10)$$

where k is any positive number sufficiently large to guarantee the existence of the integral. In the next two lemmas we give estimates for $M_k^\rho(x)$ for different ranges of x .

LEMMA 2.3.3. *Let $0 < h < 1$, $0 < \delta' < 1/3$ and $\delta'' > 3\delta'/2$. If $k^{-h} \leq x \leq k^{\delta'}$, then*

$$M_k^\rho(x) = ce^{-x} x^{\alpha\theta} k^{\alpha\theta} \cos(c'(kx)^{1/2} + \beta\pi) + O(x^{\alpha\theta} k^{\alpha\theta-1/2+\delta''}) \\ + O(x^{\alpha\theta-1/2} k^{\alpha\theta-1/2}).$$

LEMMA 2.3.4. *For δ as in (2.3.1), $x > k^\delta$, and sufficiently large k ,*

$$M_k^\rho(x) = O(x^{-(1+a)/(2\delta)}),$$

where the implied constant is absolute.

A proof of these two lemmas can be found in Berndt [2, Lemmas 3.3, 3.4]. We point out that our $M_k^\rho(x) = (2x)^{1/2} M_k(2x)$ in Berndt's notation. Also, a minor change needs to be made in Berndt's proof to accommodate the error

$$O(x^{-(1+a)/(2\alpha\delta)})$$

in the application of Lemma 2.3.2. This calculation is relatively straightforward.

We wish to use some results of Chandrasekharan and Narasimhan from [4]. To do this we must make our definition of $I_{\rho+q}$ consistent with theirs.

Thus, as in [4], define for $x > 0$,

$$I_{\rho+q}^*(x) = \frac{1}{2\pi i} \int_{c_{\rho+q}-i\infty}^{c_{\rho+q}+i\infty} G_{\rho+q}(s) x^{r+\rho+q-s} ds, \quad (2.3.11)$$

where $c_{\rho+q} = r/2 + (\rho+q)/(2\alpha) - \varepsilon_3$, with ε_3 chosen so that $0 < \varepsilon_3 < 1/(4\alpha)$ and $r - c_{\rho+q}$ is not an integer.

LEMMA 2.3.5. *For large q and $x > 0$,*

$$I_{\rho+q}(x) = I_{\rho+q}^*(x) - \sum_{\substack{v \geq 0 \\ c_\delta < r+v < c_{\rho+q}}} c_v(q) x^{\rho+q-v},$$

where the $c_v(q)$ are constants depending on v and q .

Proof. It follows from Cauchy's theorem, the analyticity of the integrand $G_{\rho+q}(s)$ in $|\mathbf{Im} s| > R$, and the absolute convergence of the integrals for $I_{\rho+q}(x)$ and $I_{\rho+q}^*(x)$ that

$$\frac{1}{2\pi i} \left\{ \int_{c_{\rho} \pm i\infty}^{c_{\rho} \pm iR} + \int_{c_{\rho} \pm iR}^{c_{\rho+q} \pm iR} + \int_{c_{\rho+q} \pm iR}^{c_{\rho+q} \pm i\infty} \right\} G_{\rho+q}(s) x^{r+\rho+q-s} ds = 0, \quad (2.3.12)$$

where we can choose either the upper or lower signs.

Furthermore, if \mathcal{R} is the rectangle with vertices $c_{\delta} \pm iR$ and $c_{\rho+q} \pm iR$, taken in the positive direction, then

$$\frac{1}{2\pi i} \int_{\mathcal{R}} G_{\rho+q}(s) x^{r+\rho+q-s} ds = \sum_{\substack{v \geq 0 \\ c_{\delta} < r+v < c_{\rho+q}}} c_v(q) x^{r+\rho+q-v}, \quad (2.3.13)$$

where

$$c_v(q) = \frac{(-1)^v \Delta(r+v)}{v! \Gamma(\rho+q+1-v) \Delta(-v)}.$$

To conclude the proof, we add the two distinct identities in (2.3.12) and subtract identity (2.3.13).

In the next lemma we give an asymptotic formula for the following integral. For $\xi > 0$ and sufficiently large k , let

$$I_{k,\xi} = \frac{1}{\Gamma(k+1)} \int_0^{\infty} e^{-u} u^k (\xi u^{\alpha})^{\theta} g(\xi u^{\alpha}) du, \quad (2.3.14)$$

where $g(x)$ is defined in (1.2.4).

LEMMA 2.3.6. *Let ϵ be a positive number less than $\alpha/2$. Then as k tends to ∞ ,*

$$I_{k,\xi} \sim c \xi^{\theta} k^{\alpha\theta} g(\xi k^{\alpha}),$$

provided $k^{-\epsilon} \leq \xi \leq \xi_0$.

Proof. We first note that conditions (iii) and (iii)' on $Z(x)$ imply that $g(x) = O(x^{\epsilon'})$ for every $\epsilon' > 0$, and condition (i) implies that $g(x^c) = O(g(x))$ for any positive constant c . Also, by our limits on ξ , we have that ξk^{α} tends to ∞ with k .

We next replace u by ku in the integral to get, after some rearrangement,

$$I_{k,\xi} = \frac{\xi^{\theta} k^{k+\alpha\theta+1} e^{-k}}{\Gamma(k+1)} \int_0^{\infty} (e^{1-u} u)^k u^{\alpha\theta} g(\xi k^{\alpha} u^{\alpha}) du. \quad (2.3.15)$$

Let I^* be the last integral in (2.3.15). Let $\frac{1}{3} < \tau < \frac{1}{2}$ and $p = k^{-\tau}$. We partition the interval $(0, \infty)$ as $(1-p, 1+p) \cup [1+p, \infty) \cup (0, 1-p]$, and label the integrals over these intervals as J_1, J_2 , and J_3 , respectively. Thus

$$I^* = J_1 + J_2 + J_3. \quad (2.3.16)$$

For u in $(1-p, 1+p)$,

$$u^{\alpha\theta} = 1 + O(p), \quad (2.3.17)$$

and

$$\begin{aligned} (e^{1-u}u)^k &= \{e^{1-u+\log(1-(1-u))}\}^k \\ &= \{\exp(-\tfrac{1}{2}(1-u)^2 + O(|1-u|^3))\}^k \\ &= \exp\{-k(1-u)^2/2\} \{1 + O(kp^3)\} \\ &= \exp\{-k(1-u)^2/2\} \{1 + o(1)\} \end{aligned} \quad (2.3.18)$$

since $kp^3 = o(1)$ as k tends to ∞ . We require a similar analysis of $g(\xi k^\alpha u^\alpha)$ for u in $(1-p, 1+p)$. As in (2.3.17), $u^\alpha = 1 + O(p)$ so that $\log \xi k^\alpha u^\alpha = \log \xi k^\alpha + O(p)$. By condition (ii) on Z ,

$$\begin{aligned} Z(|\log \xi k^\alpha u^\alpha|) &= Z(\log \xi k^\alpha + O(p)) \\ &= Z(\log \xi k^\alpha) + O(pe^{\eta \log \xi k^\alpha}) \\ &= Z(\log \xi k^\alpha) + O(pk^{\eta\alpha}). \end{aligned}$$

But $\eta < 1/(3\alpha)$ and $Z(\log \xi k^\alpha)$ tends to ∞ with k , so that

$$Z(|\log \xi k^\alpha u^\alpha|) = Z(\log \xi k^\alpha) \{1 + o(1)\}.$$

Thus,

$$\{Z(|\log \xi k^\alpha u^\alpha|)\}^\kappa = \{Z(\log \xi k^\alpha)\}^\kappa \{1 + o(1)\},$$

and for $A > 0$,

$$\{\log Z(|\log \xi k^\alpha u^\alpha|)\}^A = \{\log Z(\log \xi k^\alpha)\}^A \{1 + o(1)\}.$$

Therefore, if $\kappa > 0$, put $A = l - 1$, and if $\kappa = 0$, put $A = l$, and conclude that

$$g(\xi k^\alpha u^\alpha) = g(\xi k^\alpha) \{1 + o(1)\} \quad (2.3.19)$$

for u in $(1-p, 1+p)$.

Combining (2.3.17)–(2.3.19), we find that

$$\begin{aligned}
 J_1 &= g(\xi k^\alpha) \{1 + o(1)\} \int_{1-p}^{1+p} \exp\{-k(1-u)^2/2\} du \\
 &= 2 \sqrt{\frac{2}{k}} g(\xi k^\alpha) \{1 + o(1)\} \int_0^{p\sqrt{k/2}} e^{-v^2} dv \\
 &= ck^{-1/2} g(\xi k^\alpha) \{1 + o(1)\},
 \end{aligned} \tag{2.3.20}$$

since $pk^{1/2}$ tends to ∞ with k .

We now examine J_2 . By our initial remarks, $g(\xi k^\alpha u^\alpha) \ll \xi^{\epsilon'/\alpha} (ku)^{\epsilon'} \ll (ku)^{\epsilon'}$ for every $\epsilon' > 0$. Thus

$$J_2 \ll e^k k^{\epsilon'} \int_{1+p}^{\infty} e^{-ku} u^{k+\alpha\theta+\epsilon'} du.$$

Replace ku by $(1+p)u$ and apply Stirling's formula (B) to find that

$$\begin{aligned}
 J_2 &\ll \frac{k^{\epsilon'} e^k (1+p)^{k+\alpha\theta+1+\epsilon'}}{k^{k+\alpha\theta+1+\epsilon'}} \int_k^{\infty} e^{-(1+p)u} u^{k+\alpha\theta+\epsilon'} du \\
 &\ll \frac{k^{\epsilon'} e^k \{(1+p) e^{-p}\}^k}{k^{k+\alpha\theta+1+\epsilon'}} \int_0^{\infty} e^{-u} u^{k+\alpha\theta+\epsilon'} du \\
 &\ll k^{\epsilon'-1/2} \{(1+p) e^{-p}\}^k.
 \end{aligned} \tag{2.3.21}$$

Now with a fixed choice of upper or lower signs,

$$\begin{aligned}
 \{(1+p) e^{\mp p}\}^k &= \{e^{\mp p + \log(1 \pm p)}\}^k \\
 &= \exp\{-kp^2/2\} \{1 + o(1)\} \\
 &\ll k^{-n}
 \end{aligned} \tag{2.3.22}$$

for every $n > 0$. Thus with $n \geq \epsilon'$ and the choice of the upper sign in (2.3.22), the estimate in (2.3.21) becomes

$$J_2 \ll k^{-1/2} = o(k^{-1/2} g(\xi k^\alpha)). \tag{2.3.23}$$

Let $t = \xi^{1/\alpha} k$. For J_3 , we must distinguish between the intervals where $\xi k^\alpha u^\alpha > 1$ ($u > 1/t$) and where $0 < \xi k^\alpha u^\alpha < 1$ ($u < 1/t$). Thus write

$$\begin{aligned}
 J_3 &= \left\{ \int_0^{1/t} + \int_{1/t}^{1-p} \right\} (e^{1-u} u)^k u^{\alpha\theta} g(\xi k^\alpha u^\alpha) du \\
 &= J'_3 + J''_3,
 \end{aligned} \tag{2.3.24}$$

say. We note that by definition $g(1/x) = g(x)$. For $u > 0$, $e^{-ku} \ll 1$, so that

$$\begin{aligned} J'_3 &\ll e^k \int_0^{1/t} u^{k+\alpha\theta} g(t^\alpha u^\alpha) du \\ &= e^k \int_0^{1/t} u^{k-\alpha\theta} g((tu)^{-\alpha}) du. \end{aligned}$$

Put $v = (tu)^{-1}$, and for $\varepsilon' > 0$ and k sufficiently large to guarantee convergence,

$$\begin{aligned} J'_3 &\ll e^k t^{-k-\alpha\theta-1} \int_1^\infty v^{-k-\alpha\theta-2} g(v^\alpha) dv \\ &\ll e^k t^{-\alpha\theta-1} \int_1^\infty v^{-k-\alpha\theta-2+\varepsilon'} dv \\ &\ll e^k t^{-k-\alpha\theta-1} k^{-1}. \end{aligned}$$

But $t^{-1} = k^{-1} \xi^{-1/\alpha} \leq k^{-1/2}$ so that

$$\begin{aligned} J'_3 &\ll e^k k^{-(k+\alpha\theta+3)/2} \\ &= o(k^{-1/2} g(\xi k^\alpha)). \end{aligned} \tag{2.3.25}$$

For J''_3 , $1/t < u$, so that $\xi k^\alpha u^\alpha > 1$, and for $\varepsilon' > 0$,

$$\begin{aligned} J''_3 &= \int_{1/t}^{1-p} (e^{1-u} u)^k u^{\alpha\theta} g((tu)^\alpha) du \\ &\ll t^{\varepsilon'} e^k \int_{1/t}^{1-p} e^{-ku} u^{k+\alpha\theta+\varepsilon'} du \\ &\ll \frac{k^{\varepsilon'} e^k (1-p)^{k+\alpha\theta+1+\varepsilon'}}{k^{k+\alpha\theta+\varepsilon'+1}} \int_0^k e^{-(1-p)u} u^{k+\alpha\theta+\varepsilon'} du \\ &\ll \frac{k^{\varepsilon'} e^k \{(1-p) e^p\}^k}{k^{k+\alpha\theta+1+\varepsilon'}} \int_0^k e^{-u} u^{k+\alpha\theta+\varepsilon'} du \\ &\ll k^{-1/2} \\ &= o(k^{-1/2} g(\xi k^\alpha)), \end{aligned} \tag{2.3.26}$$

where we have used the fact that $t \ll k$, replaced ku by $(1-p)u$ in the second integral, applied (2.3.22) with $n \geq \varepsilon'$ and the lower signs, and used Stirling's formula (B).

Combining (2.3.16) with (2.3.20) and (2.3.23)–(2.3.26), we see that

$$I^* = ck^{-1/2} g(\xi k^\alpha) \{1 + o(1)\}.$$

Replacing this in (2.3.15) and applying Stirling's formula (B) one last time, we have the result as claimed.

We define two more auxiliary functions and state one final lemma. These will be used only in the proofs of Theorems B and C. For the real variable a , and $\kappa > 0$, define $h(a)$ by

$$h(a) = \int_0^\infty e^{-u^2} u^\gamma \cos(au + \beta\pi) du, \quad (2.3.27)$$

where $\gamma = 2\alpha\kappa - 1$ and β are given in (1.1.1). Also for $\xi > 0$, let

$$f(\xi, a) = \sum b(n) E(\mu_n) \mu_n^{-(\sigma_a - \kappa)} \cos(a(\xi \mu_n)^{1/(2\alpha)} + \beta\pi), \quad (2.3.28)$$

which converges for all values of a and κ .

LEMMA 2.3.7. Assume (1.2.5). If $\kappa > 0$, there exists an absolute constant c_2 so that as ξ tends to zero,

$$f(\xi, a) \sim c_2 h(a) \xi^{-\kappa} (\log 1/\xi)^{l-1}.$$

Proof. A proof of this can be found in Berndt [3, Lemma 3.1].

3. THE BOREL MEAN VALUE OF THE ERROR TERM

3.1. An Exact Formula

We prove, in this section, an exact formula for the Borel mean value of the error term $P_\rho(x)$, i.e., for

$$\frac{1}{\Gamma(k+1)} \int_0^\infty e^{-u} u^k P_\rho(2^\alpha \xi u^\alpha) du,$$

where $\xi > 0$ and k is a sufficiently large positive number.

Put $q = [2\alpha\sigma_a] + 2$. By a result of Chandrasekharan and Narasimhan [4], we have

$$P_{\rho+q}(x) = \sum b(n) \mu_n^{-r-\rho-q} I_{\rho+q}^*(x\mu_n), \quad (3.1.1)$$

where $P_{\rho+q}(x)$ and $I_{\rho+q}^*(x)$ are defined in (1.1.4) and (2.3.11), respectively. In the following, we assume the number k is large enough so that all inversions of limiting processes are justified by absolute convergence.

Set $x = 2^\alpha \xi u$ in (3.1.1), multiply this identity by

$$\frac{d^q}{du^q} \{u^{(k-\alpha+1)/\alpha} \exp(-u^{1/\alpha})\}$$

and integrate over $(0, \infty)$. Label the left hand side and the right hand side of (3.1.1) after these operations by M and N , respectively.

For M , we perform q integrations by parts, replace u by u^α and obtain

$$M = \alpha 2^{\alpha q} \xi^q \int_0^\infty e^{-u} u^k P_\rho(2^\alpha \xi u^\alpha) du. \quad (3.1.2)$$

Now, from (2.3.11) and the definition of $c_{\rho+q}$ given immediately thereafter,

$$I_{\rho+q}^*(x) = O(x^{r+\rho+q-c_{\rho+q}}).$$

Thus,

$$N \ll \int_0^\infty \frac{d^q}{dx^q} \{u^{(k-\alpha+1)/\alpha} \exp(-u^{1/\alpha})\} \\ \times \sum b(n) \mu_n^{-c_{\rho+q}} (2^\alpha \xi u)^{r+\rho+q-c_{\rho+q}} du,$$

so that since $q = [2\alpha\sigma_a] + 2 > 2\alpha\sigma_a - \alpha r - \rho + \frac{1}{2}$, we may invert the order of summation and integration in N . We do this, apply Lemma 2.3.5, integrate by parts q times and apply Lemma 2.3.1, to show that

$$N = 2^{\alpha q} \xi^q \sum \frac{b(n)}{\mu_n^{r+\rho}} \int_0^\infty u^{(k-\alpha+1)/\alpha} \exp(-u^{1/\alpha}) \\ \times \left\{ I_\rho(2^\alpha \xi u \mu_n) + \sum_{\substack{v \geq 0 \\ c_\delta < r+v < c_{\rho+q}}} c'_v(q) (2^\alpha \xi u \mu_n)^{\rho-v} \right\} du,$$

where the $c'_v(q)$ are constants depending on q and v (arising from the constants $c_v(q)$ in Lemma 2.3.5).

By the definitions of δ and c_δ in (2.3.1) and (2.3.3), respectively,

$$r+v > c_\delta = r + \rho + (1+a)/(2\alpha\delta) + \epsilon_1 \\ > r + \rho + 4|\sigma_a - r - \rho| \\ \geq \sigma_a.$$

This implies that for each $v \geq c_\delta - r$, the series $\sum b(n) \mu_n^{-r-v}$ converges absolutely, and as such represents a constant depending only on v . Thus, if we replace u by u^α and evaluate the integrals in the finite sum we find that

$$N = \alpha 2^{\alpha q} \xi^q \sum \frac{b(n)}{\mu_n^{r+\rho}} \int_0^\infty e^{-u} u^k I_\rho(2^\alpha \xi u^\alpha) du \\ + \alpha 2^{\alpha q} \xi^q \sum_{\substack{v \geq 0 \\ c_\delta < r+v < c_{\rho+q}}} (2^\alpha \xi)^{\rho-v} c''_v(q) \Gamma(k + \alpha(\rho-v) + 1),$$

where, again, the $c_v''(q)$ depend only on v and q . Equating N here with M from (3.1.2), dividing by $\Gamma(k+1)$, and applying the definition of M_k^p given in (2.3.10), we obtain the exact formula for the Borel mean value:

$$\begin{aligned} & \frac{1}{\Gamma(k+1)} \int_0^\infty e^{-u} u^k P_\rho(2^\alpha \xi u^\alpha) \\ &= \sum b(n) \mu_n^{-r-\rho} M_k^p((\xi \mu_n)^{1/\alpha}) \\ &+ \sum_{\substack{v \geq 0 \\ c_\delta < r+v < c_\rho+q}} c_v''(q) (2^\alpha \xi)^{\rho-v} \frac{\Gamma(k+\alpha(\rho-v)+1)}{\Gamma(k+1)}. \end{aligned} \quad (3.1.3)$$

3.2. An Asymptotic Formula

To develop an asymptotic formula for this Borel mean value in (3.1.3), we will use our estimates for $M_k^p(x)$ in Lemmas 2.3.3 and 2.3.4 and also estimate the finite sum trivially.

Let $0 < h < 1/2$, $\delta' = \delta$ as in (2.3.1) and $\delta'' = 1/16$. By (2.3.1), $\delta'' > (3/2)(1/25) \geq 3\delta/2 = 3\delta'/2$, as required for Lemma 2.3.3. Let $0 < \varepsilon < \alpha$. We shall assume that $k^{-h\varepsilon} \leq \xi \leq \xi_0$. We divide the series in (3.1.3) into different parts. The central part is determined by $k^{-h} \leq (\xi \mu_n)^{1/2} \leq k^\delta$ so that we may use Lemma 2.3.3 in this range. For $(\xi \mu_n)^{1/\alpha} < k^{-h}$, we have $\mu_n \leq k^{-h\alpha} \xi^{-1} \leq k^{-h(\alpha-\varepsilon)} = o(1)$ as k tends to ∞ . Thus we may assume that the sum in this range is empty. For the part determined by $(\xi \mu_n)^{1/\alpha} > k^\delta$, we may apply Lemma 2.3.4.

Throughout these calculations, we shall assume that h is taken sufficiently small to make the given estimate valid. We will also use repeatedly the definitions of δ and a given in (2.3.1).

Thus,

$$\begin{aligned} & \sum b(n) \mu_n^{-r-\rho} M_k^p((\xi \mu_n)^{1/\alpha}) \\ &= \left\{ \sum_{\mu_n \leq k^{\alpha\delta/\xi}} + \sum_{\mu_n > k^{\alpha\delta/\xi}} \right\} b(n) \mu_n^{-r-\rho} M_k^p((\xi \mu_n)^{1/\alpha}) \\ &= S_1 + S_2, \end{aligned} \quad (3.2.1)$$

say.

By Lemma 2.3.3, since $E(u) = \exp\{-\xi u\}$ and $r + \rho - \theta = \sigma_a - \kappa$,

$$S_1 = c \xi^\theta k^{\alpha\theta} \sum b(n) E(\mu_n) \mu_n^{-(\sigma_a - \kappa)} \cos(c' k^{1/2} (\xi \mu_n)^{1/(2\alpha)} + \beta\pi) + R_k, \quad (3.2.2)$$

where

$$\begin{aligned}
 R_k = & O \left(\xi^\theta k^{\alpha\theta} \sum_{\mu_n > k^{\alpha\delta}/\xi} b(n) E(\mu_n) \mu_n^{-(\sigma_a - \kappa)} \right) \\
 & + O \left(\xi^\theta k^{\alpha\theta - 7/16} \sum_{\mu_n \leq k^{\alpha\delta}/\xi} b(n) \mu_n^{-(\sigma_a - \kappa)} \right) \\
 & + O \left(\xi^{\theta - 1/(2\alpha)} k^{\alpha\theta - 1/2} \sum_{\mu_n \leq k^{\alpha\delta}/\xi} b(n) \mu_n^{-(\sigma_a - \kappa + 1/(2\alpha))} \right). \quad (3.2.3)
 \end{aligned}$$

By construction, $\xi^{-1} \leq k^{h\epsilon} \leq k^{h\alpha}$ so that $k^{\alpha\delta}/\xi \leq k^{\alpha(\delta+h)}$. By (2.2.4) and (2.2.1), with $x = k^{\alpha\delta}/\xi$ and $\epsilon' = 1/(16\alpha(\delta+h)) > 0$, the first two expressions in R_k can be estimated by

$$\begin{aligned}
 & O(\xi^{\theta - \kappa - \epsilon'} k^{\alpha\theta} \exp(-\tfrac{1}{2}k^\delta)) + O(\xi^\theta k^{\alpha\theta - 7/16} (k^{\alpha\delta}/\xi)^{\kappa + \epsilon'}) \\
 & = O(k^{\alpha h|\theta - \kappa - \epsilon'| + \alpha\theta} \exp(-\tfrac{1}{2}k^\delta)) \\
 & \quad + O(k^{\alpha\theta - 1/4} \{k^{-3/16 + \alpha h|\theta| + \alpha(\delta+h)(\kappa + \epsilon')}\}) \\
 & = O(k^{\alpha\theta - 1/4}) \quad (3.2.4)
 \end{aligned}$$

since $\delta^{-1} \geq 16\alpha\kappa$.

For the third O -term in (3.2.3), we have two possibilities. If $\kappa < 1/(2\alpha)$, then the series will converge and we can estimate this term by

$$\begin{aligned}
 & O(\xi^{\theta - 1/(2\alpha)} k^{\alpha\theta - 1/2}) \\
 & = O(k^{\alpha\theta - 1/4} \{k^{-1/4 + \alpha h|\theta - 1/(2\alpha)|}\}) \\
 & = O(k^{\alpha\theta - 1/4}). \quad (3.2.5)
 \end{aligned}$$

If $\kappa \geq 1/(2\alpha)$, then we use (2.2.1) with $x = k^{\alpha\delta}/\xi$ and $\epsilon' = 1/(2\alpha)$ to obtain the estimate

$$\begin{aligned}
 & O(\xi^{\theta - 1/(2\alpha)} k^{\alpha\theta - 1/2} (k^{\alpha\delta}/\xi)^\kappa) \\
 & = O(k^{\alpha\theta - 1/4} \{k^{-1/4 + \alpha h|\theta - 1/(2\alpha)| + \alpha(\delta+h)\kappa}\}) \\
 & = O(k^{\alpha\theta - 1/4}), \quad (3.2.5)'
 \end{aligned}$$

since $\delta^{-1} \geq 16\alpha\kappa$.

Combining (3.2.2)–(3.2.5) (or (3.2.5)' as appropriate) we have

$$\begin{aligned}
 S_1 = & c\xi^\theta k^{\alpha\theta} \sum b(n) E(\mu_n) \mu_n^{-(\sigma_a - \kappa)} \cos(c'k^{1/2}(\xi\mu_n)^{1/(2\alpha)} + \beta\pi) \\
 & + O(k^{\alpha\theta - 1/4}). \quad (3.2.6)
 \end{aligned}$$

To estimate S_2 , we apply Lemma 2.3.4. Thus, since $r + \rho + (1 + a)/(2\alpha\delta) > \sigma_a$ by (2.3.1),

$$\begin{aligned} S_2 &= O\left(\xi^{-(1+a)/(2\alpha\delta)} \sum_{\mu_n > k^{\alpha\delta/l}} b(n) \mu_n^{-r-\rho-(1+a)/(2\alpha\delta)}\right) \\ &= O((k^{\alpha\delta}/\xi)^{\sigma_a-r-\rho} (\log k)^{l-1} k^{-(1+a)/2}) \\ &= O(k^{\alpha(\delta+h)|\sigma_a-r-\rho|} (\log k)^{l-1} k^{-1/2-a/2}) \\ &= O(k^{\alpha\theta-1/4}), \end{aligned} \quad (3.2.7)$$

since $-a/2 \leq \alpha\theta$ and $8\delta\alpha|\sigma_a-r-\rho| < 1$.

We now consider the finite sum in (3.1.3). By Stirling's formula (C), this finite sum can be estimated by

$$\begin{aligned} O((\xi k^\alpha)^{r+\rho-c_\delta}) &= O(\xi^{-(1+a)/(2\alpha\delta)-\varepsilon_1} k^{-(1+a)/(2\delta)-\alpha\varepsilon_1}) \\ &= O(k^{h(1+a)/(2\delta)+h\alpha\varepsilon_1-(1+a)/(2\delta)}) \\ &= O(k^{\alpha\theta-1/4}), \end{aligned}$$

since $a \geq -2\alpha\theta$ and $\delta^{-1} > 1$.

The above estimate and those in (3.2.6) and (3.2.7) combined with (3.2.1) yield the asymptotic formula for the Borel mean value, namely,

$$\begin{aligned} &\frac{1}{\Gamma(k+1)} \int_0^\infty e^{-u} u^k P_\rho(2^\alpha \xi u^\alpha) du \\ &= c \xi^\theta k^{\alpha\theta} \sum b(n) E(\mu_n) \mu_n^{-(\sigma_a-\kappa)} \cos(c' k^{1/2} (\xi \mu_n)^{1/(2\alpha)} + \beta\pi) \\ &\quad + O(k^{\alpha\theta-1/4}). \end{aligned} \quad (3.2.8)$$

This formula is valid provided $k^{-\varepsilon} \leq \xi \leq \xi_0$, where ε is a certain fixed positive number and $\xi_0 > 0$ is sufficiently small.

4. PROOFS OF THE THEOREMS

4.1. THEOREM A

For $\kappa \geq 0$ and $\varepsilon_\rho \neq 0$, we wish to show that

$$\varepsilon_\rho \operatorname{Re} P_\rho(x) = \Omega_+(x^\theta g(x)),$$

where κ , θ , and ε_ρ are given in (1.1.1) and $g(x)$ is defined in (1.2.4).

We assume that for a suitable constant $K > 0$ and all sufficiently large x ,

$$\varepsilon_\rho \operatorname{Re} P_\rho(x) < Kx^\theta g(x). \quad (4.1.1)$$

We shall be done if we show that there exists a constant c so that

$$K > c > 0, \quad (4.1.2)$$

i.e., K cannot be taken arbitrarily small.

It follows from (4.1.1) and Lemma 2.3.6 that for sufficiently large k and $k^{-\varepsilon} \leq \xi \leq \xi_0$,

$$\begin{aligned} \frac{\varepsilon_\rho}{\Gamma(k+1)} \int_0^\infty e^{-u} u^k \operatorname{Re} P_\rho(2^\alpha \xi u^\alpha) du &\leq c + KI_{k, 2^\alpha \xi} \\ &\leq cK\xi^\theta k^{\alpha\theta} g(2^\alpha \xi k^\alpha). \end{aligned} \quad (4.1.3)$$

Combining this result with (3.2.8), we conclude that for sufficiently large k and $k^{-\varepsilon} \leq \xi \leq \xi_0$,

$$\begin{aligned} \varepsilon_\rho \sum b(n) E(\mu_n) \mu_n^{-(\sigma_a - \kappa)} \cos(c' k^{1/2} (\xi \mu_n)^{1/(2\alpha)} + \beta\pi) \\ < cKg(2^\alpha \xi k^\alpha). \end{aligned} \quad (4.1.4)$$

We make the following change of variables. Put for $K_1 > 0$,

$$\xi = \xi(t) = K_1/Z(\log t)$$

and

$$k = k(t) = (t/c')^2 \xi^{-1/\alpha}.$$

It is clear that $\xi \leq \xi_0$ if t is sufficiently large and that k tends to ∞ with t . We must show that $k^{-\varepsilon} \leq \xi$ as t tends to ∞ .

Now, by condition (iii) on Z ,

$$\begin{aligned} \log k &= 2 \log t + O(1) + O(\log Z(\log t)) \\ &= 2 \log t + o(\log t), \end{aligned}$$

so that

$$\begin{aligned} \log 1/\xi &= O(1) + \log Z(\log t) \\ &= o(\log t) \\ &= o(\log k), \end{aligned}$$

implying that $\xi \geq k^{-\varepsilon}$ for all positive ε , proving a bit more than required.

With this new notation and using condition (i) on Z , (4.1.4) can be rewritten in the form

$$\begin{aligned} \varepsilon_\rho \sum b(n) E(\mu_n) \mu_n^{-(\sigma_a - \kappa)} \cos(t\mu_n^{1/(2\alpha)} + \beta\pi) \\ < cKg(2^\alpha(t/c')^{2\alpha}) \\ < c_3Kg(t). \end{aligned} \quad (4.1.5)$$

Next let X be a large positive number and let j be an integer such that $j + \frac{1}{2}$ is nearest to β . Write $\beta = j + \frac{1}{2} + \tau$, where $0 < |\tau| \leq \frac{1}{2}$. (Clearly $\tau \neq 0$ since $\varepsilon_\rho = \cos \beta\pi \neq 0$.) Choose a positive integer q so that $4/q < |\tau|$. We apply Dirichlet's approximation theorem (Lemma 2.1.1) with $m = X$, q as above and $\gamma_n = \mu_n^{1/(2\alpha)}/(2\pi)$, where $n \leq X$ and n lies in \mathcal{P} . (The set \mathcal{P} is defined in Section 1.1.) There are at most $W(X)$ such γ 's. Thus there exists a t in the interval

$$X \leq t \leq Xq^{W(X)}$$

and integers y_n so that

$$|t\mu_n^{1/(2\alpha)} - 2\pi y_n| < 2\pi/q,$$

for $n \leq X$ and n in \mathcal{P} . With this estimate and our choice of q , we have that

$$\varepsilon_\rho \cos(t\mu_n^{1/(2\alpha)} + \beta\pi) > c_4 > 0 \quad (4.1.6)$$

for some constant c_4 .

From (4.1.5) and (4.1.6), upon some rearrangement with $c_5 = 1 + c_4$, we have

$$\begin{aligned} \left\{ c_4 \sum_{n=1}^{\infty} - c_5 \sum_{\substack{\mu_n \leq X \\ n \notin \mathcal{P}}} - c_5 \sum_{\mu_n > X} \right\} b(n) \mu_n^{-(\sigma_a - \kappa)} E(\mu_n) \\ \leq c_3Kg(t). \end{aligned} \quad (4.1.7)$$

We wish to find a lower bound for the left-hand side of this inequality. First, from Dirichlet's approximation theorem and by condition (iii) on Z ,

$$\begin{aligned} \log X \leq \log t \leq W(X) \log q + \log X \\ \leq W(X) \log q + \log Z(W(X)) \\ \leq (2 \log q) W(X) \end{aligned} \quad (4.1.8)$$

provided X is sufficiently large. Now as X tends to ∞ , so also does t . Furthermore, because of condition (i), applying the function Z to both sides of (4.1.8) yields

$$X > cZ(\log t) = cK_1/\xi. \quad (4.1.9)$$

We now return to (4.1.7). First by Lemma 2.2.4 and our choice of ξ and the definition of $g(x)$ in (1.2.4)

$$c_4 \sum_{n=1}^{\infty} b(n) \mu_n^{-(\sigma_a - \kappa)} E(\mu_n) \geq c_6 K_1^{-\kappa} g(t). \quad (4.1.10)$$

Secondly, by (2.2.7)

$$c_5 \sum_{\substack{\mu_n \leq X \\ n \notin \mathcal{P}}} b(n) \mu_n^{-(\sigma_a - \kappa)} E(\mu_n) = o(g(t)) \leq \frac{c_6}{2} K_1^{-\kappa} g(t) \quad (4.1.11)$$

provided X (and so t) is sufficiently large.

Thirdly, by Lemma 2.2.6 and (4.1.9),

$$c_5 \sum_{\mu_n > X} b(n) \mu_n^{-(\sigma_a - \kappa)} E(\mu_n) \leq c_7 K_1^{-\kappa} \exp\{-cK_1^{1/\alpha}\} g(t). \quad (4.1.12)$$

Combining (4.1.7) with (4.1.10)–(4.1.12), we conclude that

$$K_1^{-\kappa} (c_6/2 - c_7 \exp\{-cK_1^{1/\alpha}\}) < c_3 K.$$

By taking K_1 sufficiently large to make the left-hand side of this inequality positive, we have established (4.1.2) and so the proof of Theorem A is complete.

We remark that the theorem can be improved slightly. If the set \mathcal{P} is chosen so that instead of (1.1.8), we have for all sufficiently large x ,

$$B^*(x; \mathcal{P}) \geq (1 - \delta^*) B(x), \quad (4.1.13)$$

where δ^* is a small positive number, then the conclusion of the theorem remains true, provided δ^* is sufficiently small. The main adjustment in the proof is the estimate of

$$\sum_{\substack{\mu_n \leq X \\ n \notin \mathcal{P}}} b(n) \mu_n^{-(\sigma_a - \kappa)} E(\mu_n)$$

given in (4.1.11). Condition (4.1.13) translates to an estimate of type (4.1.11), where the constant depends on δ^* .

4.2. Theorem B

We assume now all the hypotheses of Theorem B. In particular, we assume that $\kappa > 0$ and that with $b = \beta - [\beta + 1/2]$ and $\gamma = 2\alpha\kappa - 1$, then either

$$2b > -\gamma \quad \text{or} \quad 2b < \gamma. \quad (4.2.1)$$

By a result of Steinig [14], $h(a)$ defined in (2.3.27) has a change of sign if and only if (4.2.1) is satisfied. Thus under this assumption, there exist constants a_1 and a_2 and positive numbers c_8, c_9, c_{10} , and c_{11} such that

$$|a_1| = c_8, \quad |a_2| = c_9, \quad h(a_1) = c_{10}, \quad h(a_2) = -c_{11}. \quad (4.2.2)$$

Let $c_{12} = c_2 \cdot \min\{c_{10}, c_{11}\}$, where c_2 is given in Lemma 2.3.7.

We will need the following estimate. If $0 < \xi \leq \xi_0$, then

$$\sum_{\substack{\mu_n \leq X \\ n \notin \mathcal{P}}} b(n) \mu_n^{-(\sigma_a - \kappa)} E(\mu_n) \leq \frac{c_{12}}{8} \xi^{-\kappa} (\log 1/\xi)^{l-1}, \quad (4.2.3)$$

uniformly in $X > 0$. This is a direct consequence of (2.2.7).

The next lemma contains the main ideas in the proofs of Theorems B and C. Recall that $f(\xi, a)$ is defined in (2.3.28).

LEMMA 4.2.1. *There exist arbitrarily large numbers t and absolute constants c_{13} and c_{14} so that if $\xi = c_{13}/Z(\log t)$, $z_1 = a_1 + t\xi^{-1/(2\alpha)}$, and $z_2 = a_2 + t\xi^{-1/(2\alpha)}$, then*

$$f(\xi, z_1) > c_{14} g(z_1)$$

and

$$-f(\xi, z_2) > c_{14} g(z_2).$$

Furthermore, z_1 and z_2 tend to ∞ with t .

Proof. Let X be a large positive number. We apply Dirichlet's approximation theorem (Lemma 2.1.1) with $q = [8\pi c_1/c_{12}] + 1$, $m = X$ and $\gamma_n = \mu_n^{1/(2\alpha)}/(2\pi)$ for $n \leq X$ and n in \mathcal{P} . Thus there exists a t in the interval

$$X \leq t \leq Xq^{w(X)}$$

and integers y_n so that for $n \leq X$, n in \mathcal{P} ,

$$|t\mu_n^{1/(2\alpha)} - 2\pi y_n| < 2\pi/q < c_{12}/(4c_1).$$

Note that t can be taken arbitrarily large by taking X large. By the mean value theorem.

$$|\cos(x+y) - \cos x| \leq |y|.$$

Thus, for all real numbers a and $\xi > 0$, and for $n \leq X$ with n in \mathcal{P} ,

$$\begin{aligned}
 & |\cos\{(a + t\xi^{-1/(2\alpha)})(\xi\mu_n)^{1/(2\alpha)} + \beta\pi\} \\
 & \quad - \cos\{a(\xi\mu_n)^{1/(2\alpha)} + \beta\pi\}| \\
 & = |\cos\{a(\xi\mu_n)^{1/(2\alpha)} + \beta\pi + (t\mu_n^{1/(2\alpha)} - 2\pi y_n)\} \\
 & \quad - \cos\{a(\xi\mu_n)^{1/(2\alpha)} + \beta\pi\}| \\
 & \leq |t\mu_n^{1/(2\alpha)} - 2\pi y_n| \\
 & < c_{12}/(4c_1).
 \end{aligned} \tag{4.2.4}$$

Put $z = a + t\xi^{-1/(2\alpha)}$. By (4.2.4), (2.2.6), (4.2.3), and (2.2.5),

$$\begin{aligned}
 & |f(\xi, z) - f(\xi, a)| \\
 & \leq \sum_{\substack{\mu_n \leq X \\ n \in \mathcal{P}}} b(n) \mu_n^{-(\sigma_a - \kappa)} E(\mu_n) |\cos\{z(\xi\mu_n)^{1/(2\alpha)} + \beta\pi\} \\
 & \quad - \cos\{a(\xi\mu_n)^{1/(2\alpha)} + \beta\pi\}| \\
 & \quad + 2 \sum_{\substack{\mu_n \leq X \\ n \notin \mathcal{P}}} b(n) \mu_n^{-(\sigma_a - \kappa)} E(\mu_n) + 2F(\xi, X) \\
 & \leq \left(\frac{c_{12}}{2} + 2c_1 E(X/2^\alpha) \right) \xi^{-\kappa} (\log 1/\xi)^{l-1}.
 \end{aligned} \tag{4.2.5}$$

By the same calculation as in (4.1.8) and (4.1.9), there exists an absolute constant c_{15} such that

$$X \geq c_{15} Z(\log t). \tag{4.2.6}$$

Let c_{13} be determined so large that

$$\frac{c_{12}}{4} - 2c_1 \exp \left\{ -\frac{1}{2} (c_{15} c_{13})^{1/\alpha} \right\} > 0 \tag{4.2.7}$$

and put $\xi = c_{13}/Z(\log t)$. By (4.2.6) and (4.2.7)

$$\frac{c_{12}}{4} - 2c_1 \exp \left\{ -\frac{1}{2} (\xi X)^{1/\alpha} \right\} = \frac{c_{12}}{4} - 2c_1 E(X/2^\alpha) > 0. \tag{4.2.8}$$

Furthermore, by Lemma 2.3.7 and the definitions of c_{10} , c_{11} , and c_{12} ,

$$\begin{aligned}
 f(\xi, a_1) & \sim c_2 h(a_1) \xi^{-\kappa} (\log 1/\xi)^{l-1} \\
 & = c_2 c_{10} \xi^{-\kappa} (\log 1/\xi)^{l-1} \\
 & \geq \frac{3}{4} c_{12} \xi^{-\kappa} (\log 1/\xi)^{l-1},
 \end{aligned} \tag{4.2.9}$$

and similarly

$$-f(\xi, a_2) \geq \frac{3}{4}c_{12}\xi^{-\kappa}(\log 1/\xi)^{l-1} \quad (4.2.10)$$

provided $0 < \xi \leq \xi_0$, i.e., t is sufficiently large.

Let z_1 and z_2 be as in the statement of the lemma. It follows from (4.2.5) and (4.2.8)–(4.2.10) and our choice of ξ that

$$\begin{aligned} f(\xi, z_1) &\geq f(\xi, a_1) - |f(\xi, z_1) - f(\xi, a_1)| \\ &\geq cg(t) \end{aligned} \quad (4.2.11)$$

and similarly

$$-f(\xi, z_2) \geq cg(t). \quad (4.2.12)$$

Now, by condition (iii)' on Z , if $z = z_1$ or z_2 ,

$$\begin{aligned} \log z &= \log t + (1/2\alpha) \log 1/\xi + O(1) \\ &= \log t + O(\log Z(\log t)) \\ &= \log t + O(\log \log t) \end{aligned} \quad (4.2.13)$$

as t tends to ∞ . Thus $t \geq z_1^{1/2}$ and $t \geq z_2^{1/2}$ if t is sufficiently large. By condition (i) on Z , (4.2.11), (4.2.12) and this last remark

$$f(\xi, z_1) > c_{14}g(z_1)$$

and

$$-f(\xi, z_2) > c_{14}g(z_2)$$

for some absolute constant c_{14} .

The estimate in (4.2.13) also shows that z_1 and z_2 tend to ∞ with t . This completes the proof of the lemma.

We can now complete the proof of Theorem B. As before, we assume there is a positive constant K such that for all sufficiently large x ,

$$\pm \operatorname{Re} P_\rho(x) < Kx^\theta g(x). \quad (4.2.14)$$

We shall be done if we show that

$$K > c_{16} > 0 \quad (4.2.15)$$

for some absolute constant c_{16} .

By the same argument as that used above to prove (4.1.4), we have, for $k^{-\epsilon} \leq \xi \leq \xi_0$, and k sufficiently large,

$$\begin{aligned} &\pm \sum b(n) E(\mu_n) \mu_n^{-(\sigma_a - \kappa)} \cos(c' k^{1/2} (\xi \mu_n)^{1/(2\alpha)} + \beta\pi) \\ &< cKg(2^\alpha \xi k^\alpha). \end{aligned}$$

Putting $z = c'k^{1/2}$, noting that $\xi \leq 1$ and that ξk^a tends to ∞ with k and using condition (i) on Z , we see that

$$\begin{aligned} \pm f(\xi, z) &\leq cKg(2^a k^a) \\ &\leq cKg(2^a (z/c')^{2a}) \\ &\leq c_{17}Kg(z), \end{aligned} \quad (4.2.16)$$

for all sufficiently large z . But with t, ξ , and $z = z_1$ or z_2 as in Lemma 4.2.1, (4.2.16) and Lemma 4.2.1 imply that

$$c_{14}g(z) < c_{17}Kg(z).$$

Clearly K cannot be taken smaller than $c_{16} = c_{14}/c_{17}$ and (4.2.15) follows. Also by Lemma 4.2.1, z_1 and z_2 can be taken arbitrarily large by an appropriate choice of t . This completes the proof of Theorem B.

4.3. THEOREM C

Let t, ξ, z_1 and z_2 be given by Lemma 4.2.1. Let $k_1 = (z_1/c')^2$ and $k_2 = (z_2/c')^2$. Then by Lemma 4.2.1, condition (i) on Z , and (3.2.8), there is a constant c_{18} so that

$$\frac{1}{\Gamma(k_1 + 1)} \int_0^\infty e^{-u} u^{k_1} \operatorname{Re} P_\rho(2^a \xi u^a) du > c_{18} \xi^\theta k_1^{a\theta} g(k_1) \quad (4.3.1)$$

and similarly

$$-\frac{1}{\Gamma(k_2 + 1)} \int_0^\infty e^{-u} u^{k_2} \operatorname{Re} P_\rho(2^a \xi u^a) du > c_{18} \xi^\theta k_2^{a\theta} g(k_2) \quad (4.3.2)$$

provided t is sufficiently large.

Since $z_1 - z_2 = a_1 - a_2$,

$$k_1 - k_2 \sim 2(a_1 - a_2) z_1 / (c')^2 \sim 2(a_1 - a_2) k_1^{1/2} / c', \quad (4.3.3)$$

as t tends to ∞ . Thus by taking t arbitrarily large according to Lemma 4.2.1, we can conclude that there are infinitely many triples (k_1, k_2, ξ) such that k_1 and k_2 are arbitrarily large, ξ satisfies

$$\frac{1}{Z(\log k)} \ll \xi \ll \frac{1}{Z(\log k)}, \quad (4.3.4)$$

where k is either k_1 or k_2 and such that (4.3.1)–(4.3.3) are satisfied. The estimate in (4.3.4) follows from an argument similar to (4.2.13).

We wish to isolate the main contributions to the integrals in (4.3.1) and

(4.3.2). To this end, let $N^+ = N^+(k) = k + \tau(k \log k)^{1/2}$ and $N^- = N^-(k) = k - \tau(k \log k)^{1/2}$, where τ is chosen so that

$$\alpha T + \alpha \rho + \frac{1}{2} - \frac{1}{2}\tau^2 < 0 \quad (4.3.5)$$

and T is as in (1.2.6). Next let I^+ and I^- be the integrals of $e^{-u}u^k P_\rho(2^\alpha \xi u^\alpha)$ over the range (N^+, ∞) and $(0, N^-)$, respectively. By Lemma 2.2.7, and the facts that $T + \rho > 0$ and $\xi \leq 1$,

$$I^+ \ll \frac{1}{\Gamma(k+1)} \int_{N^+}^{\infty} e^{-u} u^{k+\alpha T+\alpha \rho} du.$$

If k is sufficiently large so that $N^+ > k + \alpha T + \alpha \rho + 2$, then

$$\begin{aligned} I^+ &\ll \frac{e^{-N^+} (N^+)^{k+\alpha T+\alpha \rho+2}}{\Gamma(k+1)} \int_{N^+}^{\infty} u^{-2} du \\ &\ll \frac{e^{-N^+} (N^+)^{k+\alpha T+\alpha \rho+1}}{\Gamma(k+1)}. \end{aligned} \quad (4.3.6)$$

Similarly, if k is sufficiently large,

$$I^- \ll \frac{e^{-N^-} (N^-)^{k+\alpha T+\alpha \rho+1}}{\Gamma(k+1)}. \quad (4.3.7)$$

We examine (4.3.6) and (4.3.7) together. Taking logarithms, applying Stirling's formula (B), the Taylor expansion of $\log(1+x)$, and (4.3.5), we find that

$$\log I^+ \rightarrow -\infty, \quad \log I^- \rightarrow -\infty$$

as k tends to ∞ . Thus

$$I^+ = o(1), \quad I^- = o(1) \quad (4.3.8)$$

as k tends to ∞ .

A similar calculation with P_ρ replaced by 1 yields the formula

$$\frac{1}{\Gamma(k+1)} \int_N^{N^+} e^{-u} u^k du = 1 + o(1) \quad (4.3.9)$$

as k tends to ∞ .

Now let $N_1^- = N^-(k_1)$, $N_1^+ = N^+(k_1)$, $N_2^- = N^-(k_2)$, and $N_2^+ = N^+(k_2)$. From (4.3.1), (4.3.2), and (4.3.8), if k_1 (and so k_2) is sufficiently large

$$\frac{1}{\Gamma(k_1+1)} \int_{N_1^-}^{N_1^+} e^{-u} u^{k_1} \mathbf{Rc} P_\rho(2^\alpha \xi u^\alpha) du \geq \frac{c_{18}}{2} \xi^\theta k_1^{\alpha \theta} g(k_1) \quad (4.3.10)$$

and

$$-\frac{1}{\Gamma(k_2+1)} \int_{N_2^-}^{N_2} e^{-u} u^{k_2} \operatorname{Re} P_\rho(2^\alpha \xi u^\alpha) du \geq \frac{c_{18}}{2} \xi^\theta k_2^{\alpha\theta} g(k_2). \quad (4.3.11)$$

We now claim that for sufficiently large k_1 , there exists u_1 and u_2 satisfying

$$N_1^- \leq u_1 \leq N_1^+ \quad \text{and} \quad N_2^- \leq u_2 \leq N_2^+$$

such that

$$\operatorname{Re} P_\rho(2^\alpha \xi u_1^\alpha) \geq (c_{18}/4) \xi^\theta k_1^{\alpha\theta} g(k_1) \quad (4.3.12)$$

and

$$-\operatorname{Re} P_\rho(2^\alpha \xi u_2^\alpha) \geq (c_{18}/4) \xi^\theta k_2^{\alpha\theta} g(k_2). \quad (4.3.13)$$

Suppose, for example, that (4.3.12) is false. Then by (4.3.9)

$$\begin{aligned} & \frac{1}{\Gamma(k_1+1)} \int_{N_1^-}^{N_1^+} e^{-u} u^{k_1} \operatorname{Re} P_\rho(2^\alpha \xi u^\alpha) du \\ & \leq \frac{c_{18}}{4} \xi^\theta k_1^{\alpha\theta} g(k_1) \frac{1}{\Gamma(k_1+1)} \int_{N_1^-}^{N_1^+} e^{-u} u^{k_1} du \\ & < \frac{c_{18}}{2} \xi^\theta k_1^{\alpha\theta} g(k_1), \end{aligned}$$

contradicting (4.3.10). This establishes (4.3.12). Inequality (4.3.13) is established in a similar manner.

Let x_1 and x_2 be defined by

$$x_1 = 2^\alpha \xi u_1^\alpha, \quad x_2 = 2^\alpha \xi u_2^\alpha. \quad (4.3.14)$$

By conditions (i) and (iii)' on Z , the choice of u_1 and u_2 , and (4.3.4),

$$x \sim 2^\alpha \xi k^\alpha, \quad \log x \sim \alpha \log k, \quad g(k) \gg g(x), \quad (4.3.15)$$

where x and k take the values x_1 and k_1 or x_2 and k_2 .

Thus from (4.3.12)–(4.3.15), we conclude that there exists an absolute constant A such that

$$\operatorname{Re} P_\rho(x_1) > A x_1^\theta g(x_1)$$

and

$$-\operatorname{Re} P_\rho(x_2) > A x_2^\theta g(x_2).$$

To complete the proof of Theorem C, we estimate the difference $|x_1 - x_2|$. By the mean value theorem, (4.3.3), the limits on u_1 and u_2 , (4.3.4) and (4.3.15),

$$\begin{aligned} |x_1 - x_2| &\ll |\xi(u_1^\alpha - u_2^\alpha)| \\ &\ll \xi(k_1 \log k_1)^{1/2} k_1^{\alpha-1} \\ &\ll x_1^{1-1/(2\alpha)} (\log x_1)^{1/2} \{Z(\log x_1)\}^{-1/(2\alpha)} \\ &\ll x_2^{1-1/(2\alpha)} (\log x_2)^{1/2} \{Z(\log x_2)\}^{-1/(2\alpha)}. \end{aligned}$$

Writing B for the largest of the implied constants in the above inequalities and letting $y = \min\{x_1, x_2\}$, we find that

$$y \leq x_1, x_2 \leq y + By^{1-1/(2\alpha)} (\log y)^{1/2} \{Z(\log y)\}^{-1/(2\alpha)}.$$

Since x_1 and x_2 can be taken arbitrarily large, a sequence $\{y_n\}$ tending to ∞ of such y 's exists and the proof is complete.

5. EXAMPLES

5.1. Piltz Divisor Problem

Let K be an algebraic number field of degree $N = r_1 + 2r_2$ over \mathbf{Q} , the rationals, where r_1 is the number of real conjugates and $2r_2$ is the number of imaginary conjugates of K . Let $\zeta_K(s)$ denote the Dedekind zeta function associated with K and let k be a positive integer. Landau [11] showed that there exists a positive constant c depending on K and k such that $\phi(s) = c^{-s} \{\zeta_K(s)\}^k$ satisfies the functional equation

$$\phi(s) \Delta(s) = \Delta(1-s) \phi(1-s),$$

where

$$\Delta(s) = \Gamma(s/2)^{kr_1} \Gamma(s)^{kr_2}.$$

Furthermore, $\phi(s)$ is analytic in the entire s -plane except for a pole of order k at $s = 1$. The Dirichlet series for $\phi(s)$ has the form

$$\phi(s) = \sum_{\mathcal{A}} d_k(\mathcal{A}) (cN\mathcal{A})^{-s} = \sum_{n=1}^{\infty} d_k(n, K) (cn)^{-s},$$

where the first sum is taken over all the ideals in the ring of integers of K . $N\mathcal{A}$ is the norm of the ideal \mathcal{A} and $d_k(\mathcal{A})$ is the number of ways of writing

the ideal as a product of k ideals in this ring, counting order. In the second sum

$$d_k(n, K) = \sum_{\substack{\mathcal{I} \\ N\mathcal{I} = n}} d_k(\mathcal{I}).$$

If $K = \mathbf{Q}$, each ideal is principal so $\mathcal{I} = (n)$ for some integer n and $d_k(\mathcal{I}) = d_k(n, K) = d_k(n)$, the usual k -fold divisor function. In particular, $d_2(n) = d(n)$, the classical divisor function of Dirichlet.

From the remarks above, as x tends to ∞ ,

$$\sum_{\substack{\mathcal{I} \\ n\mathcal{I} \leq x}} d_k(\mathcal{I}) = \sum_{n \leq x} d_k(n, K) \sim c_{19} x (\log x)^{k-1},$$

where c_{19} is a positive constant depending on K and k .

We make the following choice for \mathcal{P} . For $A > 0$, let

$$\mathcal{P} = \mathcal{P}^A(A) = \{\mathcal{I} : d_k(\mathcal{I}) \geq (\log N\mathcal{I})^{k \log k} \exp(-A \sqrt{\log \log N\mathcal{I}})\}.$$

Then it can be shown (see Hafner [9], or for the case $K = \mathbf{Q}$ and $k = 2$, see Hafner [7, 8]) that

$$W(x) = W(x, A) = \sum_{\substack{\mathcal{I} \in \mathcal{P} \\ N\mathcal{I} \leq x}} 1 \ll x (\log x)^{k-1-k \log k} \exp(A \sqrt{\log \log x})$$

and with $A_1 = A/(2 \log k)$,

$$\sum_{\substack{\mathcal{I} \in \mathcal{P} \\ N\mathcal{I} \leq x}} d_k(\mathcal{I}) \geq c_{19} \left(1 - \frac{c}{\exp A_1}\right) x (\log x)^{k-1},$$

for sufficiently large x . Thus, we have

$$Z(x) = W^{-1}(x) \gg x (\log x)^{k \log k - k + 1} \exp(-A \sqrt{\log \log x}).$$

The error term, denoted by $\Delta_{k,\rho}(x, K)$, can be written in the form

$$\begin{aligned} \Delta_{k,\rho}(x, K) &= \Gamma(\rho + 1)^{-1} \sum_{\substack{\mathcal{I} \\ N\mathcal{I} \leq x}} d_k(\mathcal{I})(x - N\mathcal{I})^\rho \\ &\quad - \frac{1}{2\pi i} \int_{\mathcal{C}_\rho} \frac{\zeta_k^k(s) \Gamma(s) x^{s+\rho}}{\Gamma(s + \rho + 1)} ds, \end{aligned}$$

where the path \mathcal{C}_ρ is given according to (1.1.3). The first sum can also be written in the form

$$\Gamma(\rho + 1)^{-1} \sum_{n \leq x} d_k(n, K)(x - n)^\rho.$$

Put

$$G(x) = G(x, A) = x^\theta (\log x)^\kappa (\log \log x)^\tau \exp(-A \sqrt{\log \log \log x}),$$

where $\theta = (kN - 1)(2\rho + 1)/(2kN)$, $\kappa = (kN - 1 - 2\rho)/(2kN)$, and $\tau = \kappa(k \log k - k + 1) + k - 1$.

Our theorems, with the modifications indicated at the end of Section 4.1, yield the following results. There exists a positive number A depending on K , k , and ρ such that if kr_2 is an even integer, then

$$\begin{aligned} (-1)^{kr_2/2} \Delta_{k,\rho}(x, K) &= \Omega_{\pm} \{G(x)\} & \text{if } \rho < \frac{kN - 3}{2} \\ &= \Omega_{+} \{G(x)\} & \text{if } \frac{kN - 3}{2} \leq \rho < \frac{kN - 1}{2}; \end{aligned}$$

if kr_2 is odd then

$$\begin{aligned} (-1)^{(kr_2+1)/2} \Delta_{k,\rho}(x, K) \\ &= \Omega_{\pm} \{G(x)\} & \text{if } \rho < \frac{kN - 2}{2} \\ &= \Omega_{+} \{G(x)\} & \text{if } \frac{kN - 2}{2} \leq \rho < \frac{kN - 1}{2} \\ &= \Omega_{+} \{x^{(kN-1)/2} (\log \log x)^k\} & \text{if } \rho = \frac{kN - 1}{2}. \end{aligned}$$

Note that no result is obtained for $\rho = (kN - 1)/2$ when kr_2 is even. Also in the special case $\rho = 0$, we have two-sided results except in the following cases: $k = 1$ and $N = 2$ (all quadratic fields); $k = 1$ and $N = r_1 = 3$ (all totally real cubic fields); $k = 2$ and $N = 1$ (Dirichlet's divisor problem); $k = 3$ and $N = 1$ (Piltz divisor problem over \mathbf{Q} with $k = 3$).

In particular, when $K = \mathbf{Q}$ and $\rho = 0$, the classical case, we have the following result. There exists a positive constant A depending on k such that

$$\begin{aligned} \Delta_k(x) &= \Delta_{k,0}(x, \mathbf{Q}) \\ &= \Omega^* \{ (x \log x)^{(k-1)/(2k)} (\log \log x)^\tau \\ &\quad \times \exp(-A \sqrt{\log \log \log x}) \}, \end{aligned} \tag{5.1.1}$$

where $\tau = (k - 1)(k \log k - k + 1)/(2k) + k - 1$ and Ω^* is Ω_{+} if $k = 2$ or 3 and Ω_{\pm} if $k \geq 4$. This last result improves those of Hardy [10] ($k = 2$) and Szegő and Walfisz [16, 17] ($k \geq 2$), where they had $\tau = k - 1$ and $A = 0$. We mention that the author [8] had obtained the result in (5.1.1) with $k = 2$ earlier by other methods.

If K/\mathbf{Q} is Galois, then the general results above are not best possible: the τ can actually be replaced by

$$\tau = \kappa(k \log kN - k + 1) + k - 1. \quad (5.1.2)$$

We hope to give the details of this result in a forthcoming paper. It appears that (5.1.2) is the best result obtainable by this method. It is possible that more can be said if K/\mathbf{Q} is not Galois, but this has not yet been determined.

5.2. Lattice Points in Ellipsoids

Let $Q = Q(n_1, n_2, \dots, n_k) = Q(\mathbf{n})$ be a real positive definite quadratic form in $k \geq 2$ variables. For $\sigma > k/2$, the Epstein zeta function is defined by

$$\begin{aligned} \zeta(Q, s) &= \sum_{\substack{n_1, n_2, \dots, n_k = -\infty \\ \mathbf{n} \neq \mathbf{0}}}^{\infty} \{Q(\mathbf{n})\}^{-s} \\ &= \sum_{n=1}^{\infty} a(n) \lambda_n^{-s}, \end{aligned}$$

where $0 < \lambda_1 < \lambda_2 < \dots$ denote the distinct values assumed by $Q(\mathbf{n})$ and $a(n)$ denotes the multiplicity of λ_n . Also $\zeta(Q, s)$ satisfies the functional equation (see Epstein [6])

$$\pi^{-s} \Gamma(s) \zeta(Q, s) = |Q|^{-1/2} \pi^{-(k/2-s)} \Gamma(k/2-s) \zeta(Q^{-1}, k/2-s),$$

where $|Q|$ is the determinant of Q and Q^{-1} is the inverse form of Q . The function $\zeta(Q, s)$ has an analytic continuation to the whole s -plane except for a simple pole at $s = k/2$.

Let $0 < \mu_1 < \mu_2 < \dots$ denote the distinct values assumed by $Q^{-1}(\mathbf{n})$ and let $b(n)$ denote the multiplicity of μ_n . We have for some constant c_Q depending on Q ,

$$\sum_{\mu_n \leq x}' b(n) \sim c_Q x^{k/2}$$

as x tends to ∞ . Let \mathcal{S} be any set of integers such that

$$\sum_{\substack{\mu_n \leq x \\ n \in \mathcal{S}}} b(n) \sim c_Q x^{k/2}$$

and let $W(x)$ be a function satisfying

$$W(x) \geq \sum_{\substack{\mu_n \leq x \\ n \in \mathcal{S}}} 1$$

and the hypotheses of the theorems. Let $Z = W^{-1}$. We remark that if \mathcal{P} is the set of all positive integers, W (and so Z) cannot be given explicitly in general. In many specific examples, however, enough is known to do this. We give an example later.

Let $P_\rho(x, Q)$ be the appropriate error term. Then we have the following general results. If k is an even integer then

$$\begin{aligned} & (-1)^{k/2} P_\rho(x, Q) \\ &= \Omega_\pm \{x^{(k-1+2\rho)/4} (Z(\log x))^{(k-1-2\rho)/4}\} \quad \text{if } \rho < \frac{k-2}{2} \\ &= \Omega_+ \{x^{(k-1+2\rho)/4} (Z(\log x))^{(k-1-2\rho)/4}\} \quad \text{if } \frac{k-2}{2} \leq \rho < \frac{k-1}{2} \\ &= \Omega_+ \{x^{(k-1)/2} \log Z(\log x)\} \quad \text{if } \rho = \frac{k-1}{2}. \end{aligned}$$

If k is an odd integer, then

$$\begin{aligned} & (-1)^{(k-1)/2} P_\rho(x, Q) \\ &= \Omega_\pm \{x^{(k-1+2\rho)/4} (Z(\log x))^{(k-1-2\rho)/4}\} \quad \text{if } \rho < \frac{k-3}{2} \\ &= \Omega_+ \{x^{(k-1+2\rho)/4} (Z(\log x))^{(k-1-2\rho)/4}\} \quad \text{if } \frac{k-3}{2} \leq \rho < \frac{k-1}{2}. \end{aligned}$$

If $\rho = (k-1)/2$ and k is odd, we obtain no result. Note that the two-sided result for $\rho = 0$ applies only when $k \geq 4$.

When $Q = Q_0(x, y) = x^2 + y^2$, much more can be said. By a result of Landau [12] we can take $W(x) = cx/\sqrt{\log x}$ and so $Z(x) \gg x\sqrt{\log x}$. The coefficients $a(n) = b(n) = r(\lambda_n)$ are the number of ways of writing the integer λ_n as a sum of two squares. Note that $r(\lambda_n) > 0$ since λ_n runs only over those integers which can be represented as sums of two squares. Also

$$A_\rho(x) = \Gamma(\rho+1)^{-1} \sum_{n \leq x} r(n)(x-n)^\rho.$$

The above results simplify in this special case to the following:

$$\begin{aligned} P_\rho(x) &= P_\rho(x, Q_0) \\ &= \Omega_- \{x^{(2\rho+1)/4} (\log x)^{(1-2\rho)/4} (\log \log x)^{(1-2\rho)/8}\} \quad \text{if } 0 \leq \rho < \frac{1}{2} \\ &= \Omega_- \{x^{1/2} \log \log x\} \quad \text{if } \rho = \frac{1}{2} \end{aligned}$$

Note that even in the case $\rho = 0$, this improves the result of Hardy [10] by a factor of $(\log \log x)^{1/8}$.

We can refine this estimate even further by taking \mathcal{P} as

$$\mathcal{P} = \mathcal{P}(B) = \{n: r(n) \geq 4(\log n)^{\log 2} \exp(-B \sqrt{\log \log n})\}.$$

Then we can take $Z(x)$ as (see Hafner [8])

$$Z(x) = x(\log x)^{\log 2} \exp(-B \sqrt{\log \log x}).$$

Thus, there exists a constant B depending on ρ such that if $0 \leq \rho < \frac{1}{2}$,

$$P_\rho(x) = \Omega_- \{x^{(1+2\rho)/4} (\log x)^{(1-2\rho)/4} (\log \log x)^{(1-2\rho)(\log 2)/4} \\ \times \exp(-B \sqrt{\log \log \log x})\}.$$

We remark that this result is actually a special case of the result in (5.1.2) since the function $\frac{1}{4}\zeta(Q_0, s)$ is actually the Dedekind zeta function of the field $\mathbf{Q}(\sqrt{-1})$. Also, when $\rho = 0$, this result was obtained by the author [8] by other methods.

5.3. Another Divisor Function

Let k be a positive real number and let $\sigma_k(n)$ denote the sum of the k th powers of the divisors of n . We can assume $k > 0$ since $\sigma_k(n) = n^k \sigma_{-k}(n)$ and $\sigma_0(n) = d(n)$. The function $\phi(s) = \pi^{-s} \zeta(s) \zeta(s-k) = \sum \sigma_k(n) (n\pi)^{-s}$ satisfies the functional equation

$$\Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s-k}{2}\right) \phi(s) = \Gamma\left(\frac{k+1-s}{2}\right) \Gamma\left(\frac{1-s}{2}\right) \phi(k+1-s).$$

The Dirichlet series for $\phi(s)$ converges (absolutely) for $\sigma > k+1$. We take $W(x) = \pi x$ and $Z(x) = x/\pi$.

With $S_{k,\rho}(x)$ denoting the appropriate error term, we have the following results. The parameters θ and κ are given by $\theta = (2k+1+2\rho)/4$ and $\kappa = (2k+1-2\rho)/4$.

If k is an even integer, then

$$\begin{aligned} (-1)^{k/2} S_{k,\rho}(x) &= \Omega_\pm \{x^\theta (\log x)^\kappa\} & \text{if } \rho < k-1/2 \\ &= \Omega_+ \{x^\theta (\log x)^\kappa\} & \text{if } k-1/2 \leq \rho < k+1/2. \end{aligned}$$

If k is not an even integer, then

$$\begin{aligned} (-1)^{[k/2]+1} S_{k,\rho}(x) \\ &= \Omega_\pm \{x^\theta (\log x)^\kappa\} & \text{if } \rho < k/2 + [k/2] + 1/2 \\ &= \Omega_+ \{x^\theta (\log x)^\kappa\} & \text{if } k/2 + [k/2] + 1/2 \leq \rho < k+1/2 \\ &= \Omega_+ \{x^{k+1/2} \log \log x\} & \text{if } \rho = k+1/2. \end{aligned}$$

No result is obtained where $\rho = k + 1/2$ and k is an even integer.
For $\rho = 0$ and $k \leq 1/2$, we have

$$S_k(x) = S_{k,0}(x) = \Omega_{\pm} \{ (x \log x)^{(2k+1)/4} \}.$$

This improves results of Berndt [2] and Redmond [13]. Actually this result is correct if $k > 1/2$ but in this case, since $\sigma_k(n) > n^k$, we have the trivial estimate

$$S_k(x) = \Omega(x^k),$$

which supercedes our results. This later observation does not appear to have been made by either Berndt [2], Redmond [13], or Chandrasekharan and Narasimhan [4].

It does not appear that a more judicious choice of \mathcal{P} in this example exists. The function $\sigma_k(n)$ is too regularly distributed.

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